Fisher Equation for a Decaying Brane

Debashis Ghoshal
School of Physical Sciences
Jawaharlal Nehru University
New Delhi 110067, India
dghoshal@mail.jnu.ac.in

Abstract:

We consider the inhomogeneous decay of an unstable D-brane. The dynamical equation that describes this process (in light-cone time) is a variant of the non-linear reaction-diffusion equation that first made its appearance in the pioneering work of (Luther and) Fisher and appears in a variety of natural phenomena.

1 Introduction

D-branes play an important role in the dynamics of and duality relations between different string theories. These are defined by boundary conditions on open strings, but at the same time are typically stringy non-perturbative excitations of closed strings. While some D-branes carry conserved charges and are stable, others are unstable. To be precise, all D-branes of different dimensions of the bosonic string theory and more than half of those of the type I and II superstring theories are unstable. Moreover, configurations of more than one kind of D-branes could be unstable even when the constituents are stable individually; for example, a brane-anti-brane pair. A detailed understanding of this instability provides a valuable window into the behaviour of strings.

Even though a string has an infinite number of vibrational modes, the leading effects due to the instability and its characteristic features are governed by the lowest mode, the so called tachyonic scalar field. The study of unstable branes was pioneered by Sen [1], who proposed a set of precise conjectures concerning the tachyon. Since then, these conjectures have been checked in the open string field theory in the level truncation scheme, in various toy models, in the boundary string field theory, and finally, an exact solution describing the state at the (local) minimum of the tachyon potential has been constructed in open string field theory. Much less studied, however, is the time-dependent dynamical process of tachyon condensation [2–9].

Ignoring all the details and and technicalities, this is a system which has two extrema: an unstable maximum (the perturbative vacuum of *open* strings) and a stable (or locally stable, as in the case of bosonic string) minimum. It is expected that the system will make a transition from the unstable to the stable phase dynamically. This situation is ubiquitous not only in physics but in various other fields, among them biological and chemical systems. A typical equation that governs the dynamics in such cases is the Fisher equation:

$$\partial_t u(t,x) = D\partial_x^2 u(t,x) + r u(t,x)\left(1 - u(t,x)\right), \tag{1.1}$$

where, D and r are constants; (a more general function $f(u)$ may be considered in the RHS). It admits a time-dependent solution that corresponds to a *front* which separates the two phases (an unstable one at $u = 0$ and a stable phase at $u = 1$) and moves with a characteristic speed while retaining its profile. This reaction-diffusion equation and its travelling front solution has a long history: Although it was first written by Luther (1906) for a chemical system, unaware of this work, Fisher (1937) proposed this equation and studied its front solution to describe the spread of an advantageous mutation. A detailed mathematical analysis by Kolmogorov, Petrovsky and Piskunov was the first in the vast literature [10, 11] that followed. The equation of the tachyon field on an unstable D-brane turns out to be a variant of this with new elements in the form of time delay and spatial non-locality.

2 Open string field theory

Henceforth, for definiteness, we shall restrict to the unstable branes of the bosonic string theory. The tachyon equation can be obtained from the cubic open string field theory. The string field, expanded in terms of the infinite number of oscillatory modes, is

$$|\Psi\rangle = (\phi(X)\, c_1 + \cdots)\, |0\rangle = \left(\int \frac{d^n k}{(2\pi)^n} \phi(k)\, e^{ik.X}\, c_1 + \cdots \right) |0\rangle \,,$$

where ϕ denotes the scalar field corresponding to the lowest mode of the string and the dots denote the higher excitatations that have been omitted. The cubic action of the string field is of the Chern-Simons type:

$$S_{\mathrm{SFT}} = -\frac{1}{g^2} \left(\frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi \star \Psi \rangle \right) .$$

In the above, the products are defined in the conformal field theory on the upper half-plane in the usual fashion. If we retain only the tachyon field (level truncation to zeroth order) we obtain the action [2]:

$$S = -\frac{1}{g^2} \int d^n x \left[\frac{\alpha'}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \phi^2 + \frac{K^3}{3} \left(K^{\alpha' \Box} \phi \right)^3 \right] , \tag{2.1}$$

where, $K = 3\sqrt{3}/4$. The equation of motion for the tachyon

$$\alpha' \Box \phi(t, \mathbf{x}) = -\phi(t, \mathbf{x}) + K^3 e^{\alpha \Box} \left[e^{\alpha \Box} \phi(t, \mathbf{x}) \right]^2 , \tag{2.2}$$

(where, $\alpha = \alpha' \ln K$) contains an infinite number of higher derivatives in the interaction term and is, therefore, non-local.

When ϕ depends only on time (spatially homogeneous decay):

$$\frac{d^2 \phi(t)}{dt^2} = \phi(t) - K^3\, e^{-\alpha \frac{d^2}{dt^2}} \left[e^{-\alpha \frac{d^2}{dt^2}} \phi(t) \right]^2 . \tag{2.3}$$

This equation has solutions[1] that start at the maximum of the potential (at $\phi_U = 0$) towards the (local) minimum (at $\phi = K^{-3} \simeq 0.456$), but it overshoots and exhibits (non-linear) oscillations around the minimum. At late times, these behave wildly. However, there is not a solution that interpolates between the unstable and the stable extrema [2]. On the other hand, according to the conjectures of Sen, one expects the unstable D-brane to decay into some configuration of closed strings, which will carry the energy (density) of the brane. At the tree level OSFT, however, open strings do not interact with the closed string modes for this to happen.

[1]Empirically, the initial value problem turns out to be well-defined with just two initial conditions, say, the position and the velocity [2]. For further analysis of initial conditions of equations of this type, see Refs. [5, 9].

3 Branes in Linear Dilaton Background

In order to circumvent this, yet not deal with the complexities of an open-closed string field theory, one may consider open string field theory in the presence of a closed string background. Perhaps the simplest of these is a linear dilaton background[2] considered in Ref. [3]. These authors use light-cone coordinates $x^{\pm} = (t \pm x)/\sqrt{2}$, and consider the dilation profile $\mathcal{D}(x) = -D^+ x^- \equiv -bx^-$ to study the homogeneous decay of the tachyon as a function of light-cone time x^+, which we shall call τ to simplify notation. The dilaton, being linear along a null direction, changes the (world-sheet) conformal dimension of the tachyon vertex operator $e^{ik.X}$ from k^2 to $k^2 + ibk^-$ (but does not alter the matter contribution to the central charge). Consequently, the equation of motion for the tachyon gets modified from Eq.(2.2) to

$$\alpha' \left(b \frac{\partial}{\partial \tau} - \nabla_\perp^2 \right) \phi(\tau, \mathbf{x}_\perp) = \phi(\tau, \mathbf{x}_\perp) - K^3 e^{-2\alpha b \partial_\tau + \alpha \nabla_\perp^2} \left[e^{\alpha \nabla_\perp^2} \phi(\tau, \mathbf{x}_\perp) \right]^2 , \tag{3.1}$$

where $\mathbf{x}_\perp$ denotes the coordinates transverse to the light-cone coordinates. This is the 'Fisher equation for the tachyon on a decaying brane'. While there are many variants of the Fisher equation [10, 11], this particular incarnation with a nonlocal nonlinear term and a delayed dependence on time, is, to our knowledge, novel.[3]

The case where $\phi = \phi(\tau)$ depends only on time (homogeneous decay), and studied in Refs. [3, 5–9],

$$\alpha' b \, \partial_+ \phi(\tau) = \phi(\tau) - K^3 \left[\phi(\tau - 2\alpha b) \right]^2 , \tag{3.2}$$

is a canonical example of a delayed growth model used, *e.g.*, in population dynamics [10]. While the standard logistic growth model has a simple interpolating solution, the delay leads to oscillations around the stable fixed point at ϕ_S. This follows from linearizing Eq.(3.2) around the stable fixed[4] point in terms of $\phi = \phi_S + \psi$:

$$\alpha' b \partial_\tau \psi(\tau) = \psi(\tau) - \psi(\tau - 2\alpha b). \tag{3.3}$$

An ansatz on the form $\psi \sim e^{-\lambda\tau}$ leads to a transcendental equation

$$\alpha' b \lambda + 1 = 2e^{-2\alpha b \lambda}, \tag{3.4}$$

that does not have a real solution, but an infinite number of complex solutions (occurring in complex conjugate pairs). This property is characteristic of delayed differential equations. The (oscillatory) convergence to ϕ_S is determined by the solution with the smallest value of $\mathrm{Re}\,\lambda$.

[2]Another possibility that has cosmological implications is to couple the tachyon to gravity [12].

[3]Similar forms of nonlocality in interactions in biological systems [13–15] was pointed out to me by V.M. Kenkre.

[4]The linearized equation around the unstable fixed point $\phi_U = 0$, solved by $\exp(\tau/b)$, shows the system moving exponentially away from it, as one would expect of a tachyonic scalar field.

4 The Travelling Front

Getting back to the Fisher equation (3.1), let us consider only one transverse coordinate (denoted by y) for simplicity. We seek a travelling front solution that moves from the right to the left (so that at any instant of time the region to the right of the front is converging to the stable fixed point). First consider the equation linearized around $\phi_U = 0$. A trial solution of the form $\phi \sim \exp\left(k(y - v(k)\tau\right)$, leads to the dispersion relation

$$v(k) = \frac{1}{b}\left(k + \frac{1}{k}\right), \tag{4.1}$$

that has a minimum at $v_{\min} = 2/b$. The wavenumber k is real for $v(k) \geq v_{\min}$, therefore, any of these would solve the linearized equation. For the standard Fisher equation, with a large class of nonlinear interactions, the travelling front is proven to select $v_{\min}$ among this [10, 11]. This feature is likely to be true of Eq.(3.1) (with the additional elements of delay and nonlocality) because the 'initial condition'—the leading edge of the wave—is determined by the 'mass' of the tachyon. However, we shall not dwell on a more rigorous proof here.

Rather, we look for a travelling front solution in the form of $\phi(\tau, y) = \Phi(\eta = y + v\tau)$, and use singular perturbation analysis [10, 11, 16] to determine the solution $\Phi(\eta)$. Note, from Eq.(4.1), that $v^2b^2 \geq 4$. In the absence of a naturally small parameter, the idea is to scale $\eta = \sqrt{\varepsilon}\,\xi$ by $\varepsilon \equiv 1/v^2b^2 \leq 0.25$, so that the equation takes the form:

$$\partial_\xi \Phi - \frac{\varepsilon}{\sqrt{\alpha'}}\partial_\xi^2 \Phi = \sqrt{\alpha'}\Phi - \frac{1}{\sqrt{\alpha'}}K^3 \exp\left(-2\alpha\partial_\xi + \varepsilon\alpha\partial_\xi^2\right)\left[e^{\varepsilon\alpha\partial_\xi^2}\Phi\right]^2, \tag{4.2}$$

Now expand $\Phi(\xi, \varepsilon) = \sum_{n=0}^{\infty} \Phi_n(\xi)\varepsilon^n$ as a power series in ε and compare terms. The lowest order equation[5] that determines $\Phi_0(\xi)$

$$\partial_\xi \Phi_0 - \Phi_0 + K^3 e^{-2\alpha\partial_\xi}\left[\Phi_0(\xi)\right]^2 = 0, \tag{4.3}$$

is identical to the homegeneous equation (3.2). Therefore, the solution of Ref. [3] is a seed for the travelling front. The correction at $\mathcal{O}(\varepsilon)$, $\Phi_1(\xi)$ can be solved from

$$\begin{aligned} \partial_\xi \Phi_1 - \Phi_1 + 2K^3 e^{-2\alpha\partial_\xi}\left[\Phi_0\Phi_1\right] &= \\ \partial_\xi^2\Phi_0 - K^3 e^{-2\alpha\partial_\xi}&\left[4\alpha\Phi_0\partial_\xi^2\Phi_0 + 2\alpha\left(\partial_\xi\Phi_0\right)^2\right], \end{aligned} \tag{4.4}$$

after substituting $\Phi_0(\xi)$ from Eq.(4.3). The equation at $\mathcal{O}(\varepsilon^2)$ is

$$\begin{aligned} \partial_\xi \Phi_2 - \Phi_2 + 2K^3 e^{-2\alpha\partial_\xi}\left[\Phi_0\Phi_2\right] &= \\ \partial_\xi^2\Phi_1 - K^3 e^{-2\alpha\partial_\xi}&\Big[4\alpha^2\Phi_0\partial_\xi^4\Phi_0 + 8\alpha^2\partial_\xi\Phi_0\partial_\xi^3\Phi_0 + 6\alpha^2\left(\partial_\xi^2\Phi_0\right)^2 \\ &4\alpha\partial_\xi^2\Phi_0\Phi_1 + 4\alpha\partial_\xi\Phi_0\partial_\xi\Phi_1 + 4\alpha\Phi_0\partial_\xi^2\Phi_1 + \Phi_1^2\Big], \end{aligned} \tag{4.5}$$

[5] We shall set $\alpha' = 1$ from now on.

by substituting for $\Phi_0(\xi)$ and $\Phi_1(\xi)$ from Eqs.(4.3) and (4.4), respectively, and so on recursively.

The functions Φ_n for the usual Fisher equation can be found analytically. For the tachyon, the propagating front can be obtained by `NDSolve` for delayed differential equation in `mathematica`. We choose an 'initial' configuration Ae^{ξ} and adjust A such that at $\xi = 0$, $\Phi(0)$ is half-way to the stable vacuum at ϕ_S. Due to the delay, the tachyon settles to the stable vacuum after damped oscillations. For the correction at $\mathcal{O}(\varepsilon)$, to tame any unnatural behaviour of the numerical solution, we choose an 'initial configuration' for Φ_1 that is identical to the standard Fisher equation for $\xi < 0$. Indeed, in this region, the profile of the front in the two cases are rather similar. The results are displayed in Fig.1.

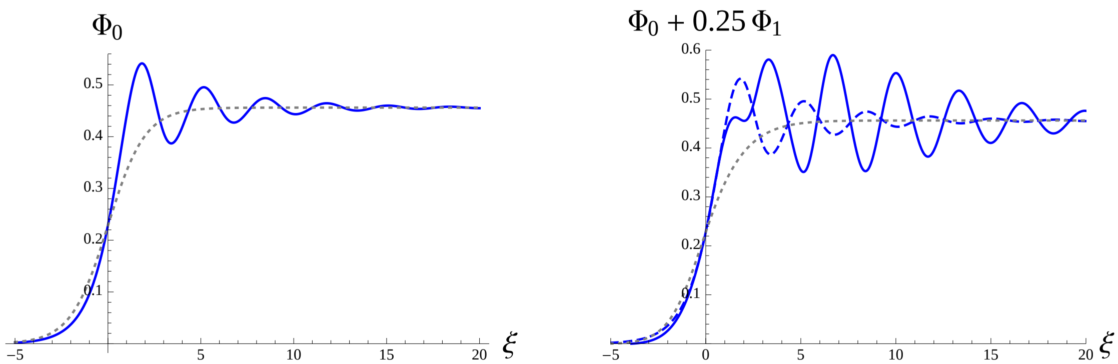

Figure 1: On the left: The leading order solutions $\Phi_0(\xi)$ of the ordinary Fisher equation (gray dotted) and the tachyon equation (4.3) (blue). On the right: Solutions upto $\mathcal{O}(\varepsilon)$—the $\mathcal{O}(1)$ solution of the tachyon equation is shown as blue dashed line. The undulation of the tachyon around the stable vacuum is characteristic of the delay.

In adapting singular perturbation theory to our problem, we have naively truncated the infinite number of derivatives in $e^{\alpha\partial_\xi^2}$ to a small finite number. One could proceed in another way to avoid this problem. We notice that the interaction term involves:

$$e^{a\partial_\xi^2} f(\xi) \equiv \mathfrak{G}[f] = \frac{1}{2\sqrt{a\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\zeta-\xi)^2}{4a}} f(\zeta) d\zeta, \tag{4.6}$$

and folding by the Gaussian kernel softens the oscillations. Since, in the limit, $a \to 0$, the kernel becomes the Dirac δ-function, $\mathfrak{d}\mathfrak{G}[\Phi_0] = \mathfrak{G}[\Phi_0] - \Phi_0$ is $\mathcal{O}(\varepsilon)$ in our case. The lowest order equation (4.3) remains unchanged, but the first order correction is now determined by:

$$\partial_\xi \Phi_1 - \Phi_1 + 2K^3 e^{-2\alpha\partial_\xi} [\Phi_0 \Phi_1] = g_1(\Phi_0), \tag{4.7}$$

where, $g_1(\Phi_0) = \partial_\xi^2 \Phi_0 - K^3 e^{-2\alpha\partial_\xi} (2\Phi_0 \mathfrak{d}\mathfrak{G}[\Phi_0] + \mathfrak{d}\mathfrak{G}[\Phi_0^2])$. Let us find Φ_1 in a different way: We begin with the standard Fisher equation, which corresponds to $\alpha = 0$. In that case, the analogue of the LHS of Eq.(4.7) (and indeed all the equations at higher orders

in ε) are simplified by the observation that $1 - 2K^3\Phi_0 = \Phi_0''/\Phi_0'$. This helps to reduce the problem of finding $\Phi_{n>0}$ to one of quadrature. Formally, this is still true as an operator equation, for a differentiation of Eq.(4.3) yields $\left[1 - 2K^3\left(e^{-2\alpha\partial_\xi}\Phi_0\right)e^{-2\alpha\partial_\xi}\right]\partial_\xi\Phi_0 = \partial_\xi^2\Phi_0$. One can now integrate in `mathematica` to find Φ_1. The constant of integration is chosen so that the correction vanishes at $\xi = 0$ [10]. The solution upto $\mathcal{O}(\varepsilon)$ obtained this way from Eqs.(4.4) and (4.7) are shown in Fig.2. Even though the error seems greater, either due to the operator identity not being too accuarte, or due to numerical integration, the singular perturbation method using the integral transform (4.7) (instead of truncation) clearly shows improved behaviour.

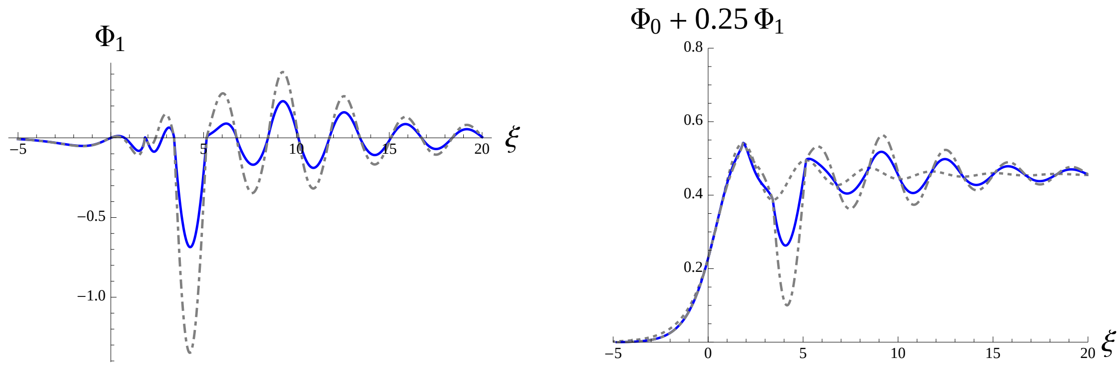

Figure 2: On the left: Solutions at $\mathcal{O}(\varepsilon)$ obtained by integration of Eqs.(4.4) (grey dot-dashed) and (4.7) (blue). On the right: Solutions upto $\mathcal{O}(\varepsilon)$. The $\mathcal{O}(1)$ solution is shown as gray dotted line.

5 Conclusions

We close with a couple of comments. The decay of an unstable brane will be triggered by the tachyon moving away from the maximum of the potential in a finite region of space. In the one dimensional case we have studied, this will lead to two fronts, one moving to the left and the other to the right. In higher dimensions, in the spherically symmetric case, the Laplacian in Eq.(3.1) in the radial variable r does not give a Fisher type equation, but will asymptote to one for large r [10].

In summary, the dynamical equation of a tachyon on an unstable D-brane is a Fisher type reaction-diffusion equation, in which the interaction is smeared by a Gaussian kernel and is also delayed. It will be interesting to see if these additional features are useful elsewhere; for example, its effect on pattern formation in biological systems [15] may be worth studying. As for the decaying brane, extension of the travelling front to a solution in string field theory as well as its stability are among the open problems.

Acknowledgments: It is a pleasure to thank Satya Majumdar (who introduced me to the Fisher equation), Ram Ramaswamy (for discussion and Ref. [10]) and Anjan Ananda Sen (for help with Mathematica). I have talked about this subject at various places—thanks are due to the hosts for hospitality and to all those in the audience for interesting questions and comments.

References

[1] A. Sen, Int. J. Mod. Phys. **A20**, 5513-5656 (2005). [hep-th/0410103] and references therein.

[2] N. Moeller, B. Zwiebach, JHEP **0210**, 034 (2002). [hep-th/0207107].

[3] S. Hellerman, M. Schnabl, [arXiv:0803.1184 [hep-th]].

[4] L. Joukovskaya, [arXiv:0803.3484 [hep-th]].

[5] N. Barnaby, N. Kamran, JHEP **0812**, 022 (2008). [arXiv:0809.4513 [hep-th]].

[6] N. Barnaby, D. Mulryne, N. Nunes, P. Robinson, JHEP **0903**, 018 (2009). [arXiv:0811.0608 [hep-th]].

[7] F. Beaujean, N. Moeller, [arXiv:0912.1232 [hep-th]].

[8] Y. Song, JHEP **1007**, 030 (2010). [arXiv:1003.0230 [hep-th]].

[9] N. Barnaby, Nucl. Phys. **B845**, 1-29 (2011). [arXiv:1005.2945 [hep-th]].

[10] J. Murray, *Mathematical Biology I: An Introduction*, Springer-Verlag, 3rd ed. (2002).

[11] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, 2e, Birkhuser, Boston (2004).

[12] N. Barnaby, T. Biswas, J. Cline, JHEP **0704**, 056 (2007). [hep-th/0612230].

[13] A. Mogliner and L. Edelstein-Keshet, J. Math. Biol. **38**, 534 (1999).

[14] C. Lee, M. Hoopes, J. Diehl, W. Gilliland, G. Huxel, V. Leaver, K. McCann, J. Umbanhowar and A. Mogliner, J. Theor. Biol. **210**, 201 (2001).

[15] M. Fuentes, M. Kuperman and V. Kenkre, Phys. Rev. Lett. **91**, 158104 (2003).

[16] S. Puri, Phys. Rev. **A43**, 7031 (1991).